

Snyder space revisited

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ABSTRACT

We examine basis functions on momentum space for the three-dimensional Euclidean Snyder algebra. We argue that the momentum space is isomorphic to the $SO(3)$ group manifold, and that the basis functions span either one of two Hilbert spaces. This implies the existence of two distinct lattice structures of space. Continuous rotations and translations are unitarily implementable on these lattices.

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1 Introduction

In 1947, Snyder wrote down a Lorentz covariant deformation of the Heisenberg algebra, with the properties that the position operators are noncommuting and have discrete spectra.[1] From the discrete position spectra, representations of the algebra imply a lattice description of space, which we refer to as ‘Snyder space’. The lattice here differs from that used to write down lattice field theories, because only one coordinate can be determined in a measurement, and the covariance indicates that the lattice is compatible with the continuous symmetry transformations of space-time.

In this article we examine representation theory for the Snyder algebra, along with the implementation of continuous transformations on the lattice. Because the time component of the space-time four vector in Snyder’s algebra has a continuous spectrum, we shall focus on the subalgebra generated by three position and momentum operators. Angular momentum operators are constructed from the position and momentum operators in the usual way, and they generate the $SO(3)$ rotation group. The group is enlarged to $SO(4)$ upon including the position operators along with the angular momentum. The discrete spectra for the position operators easily follows from the discreteness of the $SO(4)$ representations. The corresponding $so(4)$ algebra contains only one independent quadratic Casimir operator, and so unitary irreducible representations of $SO(4)$ that occur for this model are labeled by a single quantum number j , which a priori can have integer or half-integer values. The $SO(4)$ representations that result from the full Snyder algebra are infinite-dimensional reducible representations. This is due to the action of the group in momentum space. Group representations, in general, depend on the topology of the underlying space. In a related problem, the topology of momentum space played an important role in obtaining the position spectra for point particles in $2 + 1$ gravity.[2],[3] For the case of the three-dimensional Euclidean Snyder algebra, we argue that one gets a consistent action of $SO(4)$ upon making an identification at infinite momentum whereby momentum space is isomorphic to the $SO(3)$ group manifold. Because $SO(3)$ is doubly connected, the quantum theory is not unique. We find two sets of basis functions on momentum space, spanning two distinct Hilbert spaces \mathcal{H}_B and \mathcal{H}_F . The basis functions are distinguished by their asymptotic properties and also by the quantum number j , which takes all integer values for \mathcal{H}_B , and all half-integer values for \mathcal{H}_F . Because the values of j determine the spectra of the position operators, the two Hilbert spaces imply the existence of two different spatial lattices, i.e., two different Snyder spaces. Rotations and translations are implemented as unitary transformations on the lattices. In fact, the full Poincaré group (and possibly even the super-Poincaré group) can be made to have a consistent action on the lattices. This will be shown in forthcoming works by considering relativistic particle dynamics on Snyder space.[4] Nonrelativistic dynamics on three-dimensional Snyder space was recently considered in [5].

The outline for the article is the following. In section 2, we review the three-dimensional (Euclidean) Snyder algebra and obtain the spectra for the position operators. We consider

position operators associated with both Cartesian, and spherical coordinates (more specifically, we find the eigenvalues of the radial coordinate). The action of the $SO(4)$ group on momentum space is discussed in section 3. We first write down a one parameter family of differential representations for the position operators, and then impose asymptotic conditions in momentum space. We obtain basis functions associated with the Hilbert spaces \mathcal{H}_B and \mathcal{H}_F in section 4. Eigenfunctions of the radial coordinate form a convenient set of such basis functions and they correspond to spherical harmonics on S^3_+ , meaning S^3 restricted to one hemisphere. We also discuss momentum eigenfunctions of Cartesian coordinate operators. In section 5 we write down unitary transformations on the lattices associated with the translation and rotation group. Concluding remarks are made in section 6, including some speculations on the noncommutative field theory. In Appendix A we construct normalizable wavepackets in momentum space which can be explicitly transformed to the discrete position space. Finally, there are several analogies between Snyder space and the hydrogen atom, which we mention in Appendix B. In both cases, $SO(4)$ is the relevant symmetry group, there is one independent quadratic Casimir operator and the $SO(4)$ representations that appear are infinite-dimensional and reducible. On the other hand, $SO(4)$ acts differently on momentum space for the two systems.[6] Unlike the case with Snyder's algebra, momentum space for the hydrogen is topologically S^3 . The momentum-dependent eigenfunctions associated with hydrogen atom bound states were found very long ago by Podolsky and Pauling[7] and by Fock[8], and they are spherical harmonics on S^3 .

2 Snyder Algebra and Position Eigenstates

2.1 Three-dimensional Snyder algebra

The three-dimensional (Euclidean) Snyder algebra is generated by position and momentum operators, \hat{x}_i and \hat{p}_i , $i = 1, 2, 3$, respectively, which satisfies commutation relations

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= \frac{i}{\Lambda^2} \epsilon_{ijk} \hat{L}_k \\ [\hat{x}_i, \hat{p}_j] &= i \left(\delta_{ij} + \frac{\hat{p}_i \hat{p}_j}{\Lambda^2} \right) \\ [\hat{p}_i, \hat{p}_j] &= 0 \end{aligned} \tag{2.1}$$

It is a deformation of the Heisenberg algebra, where $\Lambda \neq 0$ is the deformation parameter with units of energy. The Heisenberg algebra is recovered when $\Lambda \rightarrow \infty$. $\hat{L}_i = \epsilon_{ijk} \hat{x}_j \hat{p}_k$ are the angular momenta. They satisfy the usual commutation relations

$$\begin{aligned} [\hat{x}_i, \hat{L}_j] &= i \epsilon_{ijk} \hat{x}_k \\ [\hat{p}_i, \hat{L}_j] &= i \epsilon_{ijk} \hat{p}_k \end{aligned}$$

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k, \quad (2.2)$$

and so generate the three-dimensional rotation group. In the following subsection, we enlarge to the group to $SO(4)$ by including \hat{x}_i in the set of rotation generators.

Here and throughout the article we assume that $\Lambda^2 > 0$. The case of $\Lambda^2 < 0$ is relevant for double special relativity[9] and is characterized by an upper bound on the momentum and deformed energy-momentum dispersion relations. In the latter case, \hat{x}_i and \hat{L}_i generate an $SO(3,1)$ algebra and \hat{x}_i have continuous eigenvalues. For a discussion of nonrelativistic quantum mechanics for the case $\Lambda^2 < 0$, see [5].

2.2 $so(4)$ algebra

Here we examine the $so(4)$ algebra generated by the coordinates and angular momenta, and give the spectra of the position operators. We follow the discussion in [11].

We define $\hat{L}_{AB} = -\hat{L}_{BA}$, with

$$\hat{L}_{ij} = \epsilon_{ijk}\hat{L}_k \quad \hat{L}_{i4} = \Lambda\hat{x}_i \quad (2.3)$$

Then (2.1) and (2.2) gives the standard form of the $so(4)$ algebra

$$[\hat{L}_{AB}, \hat{L}_{CD}] = i(\delta_{AC}\hat{L}_{BD} - \delta_{BC}\hat{L}_{AD} - \delta_{AD}\hat{L}_{BC} + \delta_{BD}\hat{L}_{AC}) \quad (2.4)$$

Alternatively, we have the two $SU(2)$ generators

$$\hat{A}_i = \frac{1}{2}(\hat{L}_i + \Lambda\hat{x}_i) \quad \hat{B}_i = \frac{1}{2}(\hat{L}_i - \Lambda\hat{x}_i), \quad (2.5)$$

satisfying

$$\begin{aligned} [\hat{A}_i, \hat{A}_j] &= i\epsilon_{ijk}\hat{A}_k \\ [\hat{B}_i, \hat{B}_j] &= i\epsilon_{ijk}\hat{B}_k \\ [\hat{A}_i, \hat{B}_j] &= 0 \end{aligned} \quad (2.6)$$

From (2.2) one has the identity

$$\hat{x}_i\hat{L}_i = \hat{L}_i\hat{x}_i = 0, \quad (2.7)$$

which implies

$$\hat{A}_i\hat{A}_i = \hat{B}_i\hat{B}_i \quad (2.8)$$

and hence there is only one independent quadratic Casimir operator for $so(4)$.

$\hat{A}_i\hat{A}_i$, \hat{A}_3 and \hat{B}_3 form a complete set of commuting operators, and so we can write down the three independent eigenvalue equations

$$\hat{A}_i\hat{A}_i |j, m_A, m_B\rangle = j(j+1) |j, m_A, m_B\rangle$$

$$\hat{A}_3 |j, m_A, m_B\rangle = m_A |j, m_A, m_B\rangle$$

$$\hat{B}_3 |j, m_A, m_B\rangle = m_B |j, m_A, m_B\rangle, \quad (2.9)$$

where $m_A, m_B = -j, 1-j, \dots, j$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. Since there is only one independent Casimir operator, we need only one index j to label the irreducible representations of the $so(4)$ algebra. The representations of the Snyder algebra are reducible representations of $so(4)$ because \hat{p}_i does not commute with $\hat{A}_i \hat{A}_i$. The values of j which occur in a given representation of the Snyder algebra are given in the next section. The set of all eigenvalues $\{|j, m_A, m_B\rangle\}$ are also eigenstates of \hat{L}_3 and \hat{x}_3 ,

$$\begin{aligned} \hat{L}_3 |j, m_A, m_B\rangle &= (m_A + m_B) |j, m_A, m_B\rangle \\ \hat{x}_3 |j, m_A, m_B\rangle &= \frac{1}{\Lambda} (m_A - m_B) |j, m_A, m_B\rangle \end{aligned} \quad (2.10)$$

The eigenvalues for \hat{x}_3 are evenly spaced. The result also holds for \hat{x}_1 and \hat{x}_2 , or any choice of Cartesian coordinates. Consequently, Snyder space corresponds to a cubical lattice with lattice size Λ^{-1} , where no two directions are simultaneously measurable. The eigenvalues measured in any particular direction are infinitely degenerate. From (2.10), the set of degenerate eigenvectors associated with any particular eigenvalue n_3/Λ , $n_3 = \text{integer}$, of \hat{x}_3 is

$$\left\{ |j, k, k - n_3\rangle, \quad k \in \text{integers}, \quad j \geq |k| \text{ and } |k - n_3| \right\} \quad (2.11)$$

It is not necessary to restrict to measurements along Cartesian directions. For example, by utilizing another basis for $so(4)$, we can easily obtain the spectra for the radial coordinate. This basis is associated with the sum of the two $SU(2)$ generators $\hat{A}_i + \hat{B}_i$, which is just the orbital angular momentum L_i . The basis, which we denote by $\{|j, \ell, m\rangle_\circ\}$, diagonalizes $\hat{A}_i \hat{A}_i$, $\hat{L}_i \hat{L}_i$ and \hat{L}_3 , i.e.,

$$\begin{aligned} \hat{L}_i \hat{L}_i |j, \ell, m\rangle_\circ &= \ell(\ell + 1) |j, \ell, m\rangle_\circ \\ \hat{L}_3 |j, \ell, m\rangle_\circ &= m |j, \ell, m\rangle_\circ, \end{aligned} \quad (2.12)$$

ℓ and m taking integer values, $\ell = 2j, 2j - 1, \dots, 1, 0$, and $m = m_A + m_B = -\ell, -\ell + 1, \dots, \ell$. $j(j + 1)$ is once again the eigenvalue of $\hat{A}_i \hat{A}_i$. Of course, we also have

$$(\hat{L}_1 \pm i\hat{L}_2) |j, \ell, m\rangle_\circ = \sqrt{\ell(\ell + 1) - m(m \pm 1)} |j, \ell, m \pm 1\rangle_\circ. \quad (2.13)$$

The basis vectors $|j, \ell, m\rangle_\circ$ are also eigenvectors of $\hat{x}_i \hat{x}_i$:

$$\hat{x}_i \hat{x}_i |j, \ell, m\rangle_\circ = \frac{4j(j + 1) - \ell(\ell + 1)}{\Lambda^2} |j, \ell, m\rangle_\circ, \quad (2.14)$$

and so the radial coordinate takes the values $\sqrt{4j(j + 1) - \ell(\ell + 1)}/\Lambda$. For a given j , it ranges from $\sqrt{2j}/\Lambda$ to $2\sqrt{j(j + 1)}/\Lambda$ and the eigenvalues are $2\ell + 1$ degenerate. Snyder space in this basis correspond to a set of concentric spheres, and as was observed in [10], the area of the spheres is quantized.

3 Momentum space

3.1 Differential representations

In the previous section, we only examined the algebra generated by \hat{x}_i and \hat{L}_i . Here we include the momentum operators \hat{p}_i . The latter are simultaneously diagonalizable. We denote their eigenvalues and eigenvectors by p_i and $|\vec{p}\rangle$, respectively,

$$\hat{p}_i |\vec{p}\rangle = p_i |\vec{p}\rangle \quad (3.1)$$

The eigenvalues are continuous, and the set of all of them defines momentum space. In what follows, we write down differential representations of the $so(4)$ algebra on momentum space. One such representation was given in Snyder's work[1]. Here we extend Snyder's result to a one parameter family of differential representations. An alternative representation is obtained by taking the Fourier transform, which we briefly comment on at the end of this section.

We proceed by first writing down the deformation map from the Heisenberg algebra to the Snyder algebra. The former is generated by \hat{q}_i and \hat{p}_i , satisfying

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij} \quad [\hat{q}_i, \hat{q}_j] = 0, \quad (3.2)$$

and of course $[\hat{p}_i, \hat{p}_j] = 0$. The map relates \hat{q}_i to Snyder's position operators \hat{x}_i , and is given by

$$\hat{x}_i = \hat{q}_i + \frac{1}{2\Lambda^2} (\hat{p}_i \hat{q}_j \hat{p}_j + \hat{p}_j \hat{q}_j \hat{p}_i), \quad (3.3)$$

while the momentum operators are unchanged by the map.

In the momentum representation, we replace \hat{q}_i by $i\frac{\partial}{\partial p_i}$ and so the operators \hat{x}_i and \hat{L}_i can be realized by

$$\hat{x}_i \rightarrow i\frac{\partial}{\partial p_i} + \frac{ip_i}{\Lambda^2} \left(p_j \frac{\partial}{\partial p_j} + 2 \right) \quad (3.4)$$

$$\hat{L}_i \rightarrow -i\epsilon_{ijk} p_j \frac{\partial}{\partial p_k}, \quad (3.5)$$

which act on functions $\psi(\vec{p}) = \langle \vec{p} | \psi \rangle$, where $|\psi\rangle$ is a vector in the Hilbert space for the Snyder algebra and we are using Dirac notation. From (3.5), the result found earlier that ℓ is an integer means that wavefunctions in momentum space are single-valued. The operators are symmetric for the scalar product $\langle \phi | \psi \rangle = \int d^3p \phi(\vec{p})^* \psi(\vec{p})$, provided the functions ϕ and ψ vanish sufficiently rapidly at spatial infinity in momentum space.

More generally, we can preserve the Snyder algebra by adding a term proportional to \hat{p}_i in (3.3). So we can generalize the differential operator (3.4) to

$$\hat{x}_i \rightarrow i\frac{\partial}{\partial p_i} + \frac{ip_i}{\Lambda^2} \left(p_j \frac{\partial}{\partial p_j} + \alpha \right), \quad (3.6)$$

where we restrict α to the reals. A value of α different from 2 deforms the integration measure from d^3p to

$$d\mu(\vec{p}) = \frac{d^3p}{(1 + \frac{p^2}{\Lambda^2})^{2-\alpha}} \quad (3.7)$$

The operators are now symmetric for the scalar product

$$\langle \phi | \psi \rangle = \int d\mu(\vec{p}) \phi(\vec{p})^* \psi(\vec{p}) , \quad (3.8)$$

for functions ϕ and ψ satisfying asymptotic conditions which depend on α . $\int d\mu(\vec{p}) |\vec{p}\rangle \langle \vec{p}|$ is the identity operator on the Hilbert space. For $\psi(\vec{p})$ to be normalizable it must go like $1/|\vec{p}|^w$, as $|\vec{p}| \rightarrow \infty$, where $w > \alpha - \frac{1}{2}$. We recover the trivial measure for $\alpha = 2$, while $\alpha = 0$ was the choice of Snyder[1].[‡] For $\alpha = 0$, normalizable functions $\psi(\vec{p})$ need not vanish as $|\vec{p}| \rightarrow \infty$.

Although in this article we shall rely exclusively on representations in momentum space, we here mention representations on the Fourier transform space. The latter is spanned by functions $\tilde{\psi}(\vec{q}) = \langle \vec{q} | \psi \rangle$, q_i and $|\vec{q}\rangle$, respectively, denoting the eigenvalues and eigenvectors of \hat{q}_i , satisfying (3.2). Here we represent \hat{p}_i by $-i\frac{\partial}{\partial q_i}$, and so from (3.3), \hat{x}_i involves second order derivatives. \hat{x}_i and \hat{L}_i are realized by[§]

$$\begin{aligned} \hat{x}_i &\rightarrow q_i - \frac{1}{\Lambda^2} \left(2 \frac{\partial}{\partial q_i} + q_j \frac{\partial^2}{\partial q_i \partial q_j} \right) \\ \hat{L}_i &\rightarrow -i\epsilon_{ijk} q_j \frac{\partial}{\partial q_k} \end{aligned} \quad (3.9)$$

The differential operators are symmetric for the scalar product which utilizes the trivial measure d^3q ; i.e., the scalar product between two functions $\tilde{\phi}$ and $\tilde{\psi}$ is $\langle \tilde{\phi} | \tilde{\psi} \rangle = \int d^3q \tilde{\phi}(\vec{q})^* \tilde{\psi}(\vec{q})$.

3.2 Asymptotic conditions

Here we argue that in order to define a consistent action of $SO(4)$, it is necessary to impose certain asymptotic conditions in momentum space. The result is that momentum space is isomorphic to the $SO(3)$ group manifold. For this, we first write down the following map from the three-momentum to four operators \hat{P}_A , $A = 1, 2, 3, 4$, according to

$$\hat{P}_i = \frac{\hat{p}_i}{\sqrt{\hat{p}^2 + \Lambda^2}} \quad \hat{P}_4 = \frac{\Lambda}{\sqrt{\hat{p}^2 + \Lambda^2}} \quad (3.10)$$

Recalling that \hat{L}_{AB} in (2.3) and (2.4) are the $SO(4)$ generators, one then finds that \hat{P}_A rotates as a four vector,

$$[\hat{L}_{AB}, \hat{P}_C] = i(\delta_{AC} \hat{P}_B - \delta_{BC} \hat{P}_A) \quad (3.11)$$

[‡]The $\alpha = 0$ differential representation in [1] was written down for the four-dimensional Minkowski version of (2.1), and the measure that appears there is $d^4p/(1 + \frac{p^\mu p_\mu}{\Lambda^2})^{5/2}$.

[§]A similar representation was found in [12], however the differential operators there were not symmetric.

The four momentum operators are constrained by

$$\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2 + \hat{P}_4^2 = \mathbb{1} , \quad (3.12)$$

where $\mathbb{1}$ is the identity, and so their eigenvalues lie on S^3 . It corresponds to a slice of the four-dimensional de Sitter space discussed in [1]. The eigenvalues of \hat{P}_A do not span all of S^3 . Choosing $\Lambda > 0$, \hat{P}_4 has positive-definite eigenvalues, and so the coordinatization (3.10) gives a restriction to one hemisphere of S^3 , while $\Lambda < 0$ gives a restriction to the other hemisphere of S^3 .

Equivalently, one can define a Λ -dependent map $g^{(\Lambda)}$ from $\mathbb{R}^3 = \{\vec{p}\}$ to $SU(2)$ according to

$$g^{(\Lambda)}(\vec{p}) = \frac{1}{\sqrt{\vec{p}^2 + \Lambda^2}} \begin{pmatrix} \Lambda + ip_3 & p_2 + ip_1 \\ -p_2 + ip_1 & \Lambda - ip_3 \end{pmatrix} \in SU(2) , \quad (3.13)$$

where, as before, p_i denotes the eigenvalues of \hat{p}_i . Applying the differential representations (3.5) and (3.6), one gets

$$\begin{aligned} \hat{L}_i g^{(\Lambda)}(\vec{p}) &= -\frac{1}{2} [\sigma_i, g^{(\Lambda)}(\vec{p})] \\ \hat{x}_i g^{(\Lambda)}(\vec{p}) &= -\frac{1}{2\Lambda} [\sigma_i, g^{(\Lambda)}(\vec{p})]_+ , \end{aligned} \quad (3.14)$$

where σ_i are the Pauli matrices, $[\ , \]_+$ denotes the anticommutator and we have chosen $\alpha = 0$ for convenience. It follows that \hat{A}_i and \hat{B}_i act, respectively, as left and right generators of $SU(2)$,

$$\begin{aligned} \hat{A}_i g^{(\Lambda)}(\vec{p}) &= -\frac{1}{2} \sigma_i g^{(\Lambda)}(\vec{p}) \\ \hat{B}_i g^{(\Lambda)}(\vec{p}) &= \frac{1}{2} g^{(\Lambda)}(\vec{p}) \sigma_i \end{aligned} \quad (3.15)$$

The map (3.13) when applied to all of momentum space, $g^{(\Lambda)} \circ \mathbb{R}^3$, does not cover all of $SU(2)$. Rather, assuming $\Lambda > 0$, it is a restriction to $SU(2)$ matrices satisfying

$$\text{Re } g^{(\Lambda)}(\vec{p})_{11} > 0, \quad \text{Re } g^{(\Lambda)}(\vec{p})_{22} > 0 \quad (3.16)$$

Furthermore, $g^{(\Lambda)} \circ \mathbb{R}^3$, for fixed Λ , is not invariant under $SO(4)$ because points on one hemisphere of S^3 can be rotated to the opposite hemisphere. That is, the conditions (3.16) are not preserved under $SO(4)$. Thus the action of \hat{A}_i and \hat{B}_i in (3.15) cannot be consistently exponentiated.

Let us next introduce the complementary map $g^{(-\Lambda)}$ from $\mathbb{R}^3 = \{\vec{p}\}$ to $SU(2)$. It gives a restriction to $SU(2)$ matrices satisfying

$$\text{Re } g^{(-\Lambda)}(\vec{p})_{11} < 0, \quad \text{Re } g^{(-\Lambda)}(\vec{p})_{22} < 0 \quad (3.17)$$

Then $[g^{(\Lambda)} \circ \mathbb{R}^3] \cup [g^{(-\Lambda)} \circ \mathbb{R}^3]$ spans all of $SU(2)$ and is invariant under the action of $SO(4)$. For finite momentum, $|\vec{p}| < \infty$, the two maps identify each point \vec{p} in \mathbb{R}^3 with two points in $SU(2)$. Using

$$g^{(-\Lambda)}(-\vec{p}) = -g^{(\Lambda)}(\vec{p}) , \quad (3.18)$$

the two maps are related by $Z_2 = \{1, -1\}$. There is thus a $2-1$ map from $SU(2)$ to $\{\vec{p}, |\vec{p}| < \infty\}$. The restriction to finite momentum can be lifted upon imposing appropriate asymptotic conditions in momentum space. For (3.18) to hold as $|\vec{p}| \rightarrow \infty$, we need to identify opposite points at infinity,

$$\vec{p} \leftrightarrow -\vec{p}, \quad \text{as} \quad |\vec{p}| \rightarrow \infty \quad (3.19)$$

These asymptotic conditions mean that momentum space is $SU(2)/Z_2 = SO(3)$. The $SO(3)$ matrices $\{R_{ij}(\vec{p})\}$ are given explicitly by

$$\sigma_i R_{ij}(\vec{p}) = g^{(\Lambda)}(\vec{p}) \sigma_j g^{(\Lambda)}(\vec{p})^\dagger \quad (3.20)$$

Upon applying (3.13),

$$R(\vec{p}) = \frac{1}{p^2 + \Lambda^2} \begin{pmatrix} \Lambda^2 + p_1^2 - p_2^2 - p_3^2 & 2(p_1 p_2 + \Lambda p_3) & 2(p_1 p_3 - \Lambda p_2) \\ 2(p_1 p_2 - \Lambda p_3) & \Lambda^2 - p_1^2 + p_2^2 - p_3^2 & 2(\Lambda p_1 + p_2 p_3) \\ 2(\Lambda p_2 + p_1 p_3) & 2(p_2 p_3 - \Lambda p_1) & \Lambda^2 - p_1^2 - p_2^2 + p_3^2 \end{pmatrix} \quad (3.21)$$

From (3.14), the action of $SO(4)$ generators on these matrices is given by

$$\begin{aligned} \hat{L}_i R_{jk}(\vec{p}) &= -[T_i, R(\vec{p})]_{jk} \\ \hat{x}_i R_{jk}(\vec{p}) &= -\frac{1}{\Lambda} \left([T_i, R(\vec{p})]_+ \right)_{jk}, \quad (T_i)_{jk} = -i \epsilon_{ijk}, \end{aligned} \quad (3.22)$$

where we have again chosen $\alpha = 0$ for convenience. Equivalently, \hat{A}_i and \hat{B}_i are, respectively, the left and right generators of $SO(3)$,

$$\begin{aligned} \hat{A}_i R_{jk}(\vec{p}) &= -[T_i R(\vec{p})]_{jk} \\ \hat{B}_i R_{jk}(\vec{p}) &= [R(\vec{p}) T_i]_{jk} \end{aligned} \quad (3.23)$$

This action can be consistently exponentiated to $SO(3) \times SO(3)$ acting on momentum space by left and right multiplication.

One can promote R_{ij} to operator-valued matrix elements \hat{R}_{ij} . For this we only need to replace p_i by the operators \hat{p}_i in the definition of the $SO(3)$ matrices in (3.20). The result is the set of operators $\hat{R}_{ij} = R_{ij}(\hat{\vec{p}})$, whose eigenvalues are $R_{ij}(\vec{p})$

$$\hat{R}_{ij} |\vec{p}\rangle = R_{ij}(\vec{p}) |\vec{p}\rangle \quad (3.24)$$

With the imposition of the asymptotic conditions (3.19) on momentum space, the Snyder algebra is generated by \hat{A}_i , \hat{B}_i and \hat{R}_{ij} . They satisfy commutation relations (2.6), along with

$$[\hat{A}_i, \hat{R}_{jk}] = -[T_i \hat{R}]_{jk}$$

$$[\hat{B}_i, \hat{R}_{jk}] = [\hat{R} T_i]_{jk}$$

$$[\hat{R}_{ij}, \hat{R}_{k\ell}] = 0 \quad (3.25)$$

This, along with (2.6), is an alternative way to write the three-dimensional Euclidean Snyder algebra, which takes into account the topology of momentum space. In the next section we look at the different representations of this algebra.

4 Wavefunctions on momentum space

Here we find two distinct representations of the three-dimensional Euclidean Snyder algebra. We examine them using three different bases in the subsections that follow. The first deals directly with the algebra (2.6) and (3.25), the second are momentum eigenfunctions of the radial coordinate $\sqrt{\hat{x}_i \hat{x}_i}$ and the third are momentum eigenfunctions of \hat{x}_3 .

4.1 Two Hilbert spaces

Multiple connectivity in a classical theory implies that there are multiple quantizations of the system.[13] We can view (2.6) and (3.25) as resulting from quantization on a doubly connected momentum space. This leads to two distinct representations. We denote the corresponding Hilbert spaces by \mathcal{H}_B and \mathcal{H}_F . They are as follows:

The Hilbert space \mathcal{H}_B , consists of complex functions $\{\Phi_B, \Psi_B, \dots\}$ on $SO(3) = \{R\}$. The scalar product between any two such functions Φ_B and Ψ_B is an integral over $SO(3)$,

$$\langle \Phi_B | \Psi_B \rangle = \int_{SO(3)} d\mu(R) \Phi_B(R)^* \Psi_B(R), \quad (4.1)$$

where $d\mu(R)_{SO(3)}$ is the invariant measure on $SO(3)$. From the Peter-Weyl theorem, any function Ψ_B in \mathcal{H}_B can be expanded in terms of the $(2j_B + 1) \times (2j_B + 1)$ irreducible matrix representations $\{D^{j_B}(R), j_B = 0, 1, 2, \dots\}$ of $SO(3)$, which serve as basis functions,

$$\Psi_B(R) = \sum_{j_B=0}^{\infty} \sum_{m,n=-j_B}^{j_B} a_{mn}^{j_B} D_{mn}^{j_B}(R), \quad (4.2)$$

$a_{mn}^{j_B}$ are constants. \hat{A}_i and \hat{B}_i act on the irreducible matrix representations according to

$$\begin{aligned} \hat{A}_i D_{mn}^{j_B}(R) &= -[D^{j_B}(T_i) D^{j_B}(R)]_{mn} \\ \hat{B}_i D_{mn}^{j_B}(R) &= [D^{j_B}(R) D^{j_B}(T_i)]_{mn}, \end{aligned} \quad (4.3)$$

where $D^{j_B}(T_i)$ denote the $(2j_B + 1) \times (2j_B + 1)$ matrix representations of the $SO(3)$ generators. $\hat{R}_{ij} D_{mn}^{j_B}(R)$, for $j_B \geq 1$, is a linear combination of $D^{j_B+1}(R)$, $D^{j_B}(R)$ and $D^{j_B-1}(R)$. Applying the Casimir operator, one gets

$$\hat{A}_i \hat{A}_i D_{mn}^{j_B}(R) = \hat{B}_i \hat{B}_i D_{mn}^{j_B}(R) = j_B(j_B + 1) D_{mn}^{j_B}(R), \quad (4.4)$$

and so j appearing in (2.9) belongs to the set of all integers for the Hilbert space \mathcal{H}_B .

The Hilbert space \mathcal{H}_F , consists of complex functions $\{\Phi_F, \Psi_F, \dots\}$ on $SU(2) = \{g\}$, with scalar product

$$\langle \Phi_F | \Psi_F \rangle = \int_{SU(2)} d\mu(g) \Phi_F(g)^* \Psi_F(g) , \quad (4.5)$$

where $d\mu(g)_{SU(2)}$ is the invariant measure on $SU(2)$. The basis functions for \mathcal{H}_F are restricted to all half-integer irreducible matrix representations of $SU(2)$, $\{\mathbb{D}^{j_F}(g), j_F = \frac{1}{2}, \frac{3}{2}, \dots\}$. Thus any Ψ_F in \mathcal{H}_F has the expansion

$$\Psi_F(g) = \sum_{j_F = \frac{1}{2}, \frac{3}{2}, \dots} \sum_{m, n = -j_F}^{j_F} a_{mn}^{j_F} \mathbb{D}_{mn}^{j_F}(g) , \quad (4.6)$$

$a_{mn}^{j_F}$ are constants. \hat{A}_i and \hat{B}_i act on the irreducible matrix representations according to

$$\begin{aligned} \hat{A}_i \mathbb{D}_{mn}^{j_F}(g) &= - [\mathbb{D}^{j_F}(T_i) \mathbb{D}^{j_F}(g)]_{mn} \\ \hat{B}_i \mathbb{D}_{mn}^{j_F}(g) &= [\mathbb{D}^{j_F}(g) \mathbb{D}^{j_F}(T_i)]_{mn} , \end{aligned} \quad (4.7)$$

where $\mathbb{D}^{j_F}(T_i)$ denote the $(2j_F + 1) \times (2j_F + 1)$ matrix representations of the $SU(2)$ generators. Now

$$\hat{A}_i \hat{A}_i \mathbb{D}_{mn}^{j_F}(g) = \hat{B}_i \hat{B}_i \mathbb{D}_{mn}^{j_F}(g) = j_F(j_F + 1) \mathbb{D}_{mn}^{j_F}(g) , \quad (4.8)$$

and only half-integer values of $j = j_F$ occur in (2.9) for the Hilbert space \mathcal{H}_F .

4.2 Eigenfunctions of $\hat{x}_i \hat{x}_i$, $\hat{L}_i \hat{L}_i$ and \hat{L}_3

Here we write down the momentum-dependent eigenfunctions $\phi_{j,\ell,m}(\vec{p}) = \langle \vec{p} | j, \ell, m \rangle$ of $\hat{x}_i \hat{x}_i$, $\hat{L}_i \hat{L}_i$ and \hat{L}_3 . (An alternative discussion of momentum eigenfunctions of the radial coordinate can be found in [5].) We show that these eigenfunctions are related to the spherical harmonics of S^3 - restricted to one hemisphere S^3_+ . As in the previous subsection, we find two distinct Hilbert spaces, one consisting of $\phi_{j,\ell,m}(\vec{p})$ with j integer and the other j half-integer. Here we do not make any initial assumptions on the domain of the wavefunctions, such as (3.19). For generality, we drop the restriction to $\alpha = 0$, which was made in the previous subsection. Using (3.6), the differential representation of $\hat{x}_i \hat{x}_i$ is given by

$$-\Lambda^2 \hat{x}_i \hat{x}_i \rightarrow -\frac{\hat{L}_i \hat{L}_i}{\rho^2} + (1 + \rho^2)^2 \left(\partial_\rho^2 + \frac{2}{\rho} \partial_\rho \right) + \alpha \left[2(1 + \rho^2) \rho \partial_\rho + (1 + \alpha) \rho^2 + 3 \right] , \quad (4.9)$$

where ρ is the rescaled radial component of the momentum, $\rho = |\vec{p}|/\Lambda$. Its eigenfunctions are proportional to the spherical harmonics $Y_m^\ell(\theta, \phi)$ on S^2 ,

$$\phi_{j,\ell,m}(\vec{p}) = \frac{1}{\rho} u_{j,l}(\rho) Y_m^\ell(\theta, \phi) , \quad (4.10)$$

where θ and ϕ are spherical angles in momentum space. From the eigenvalue equation (2.14) we get the following differential equation for the function $u_{j,\ell}(\rho)$

$$\left\{ (1 + \rho^2) \frac{\partial^2}{\partial \rho^2} + 2\alpha\rho \frac{\partial}{\partial \rho} + \frac{4j(j+1) + \alpha(1 + \rho^2(\alpha - 1))}{1 + \rho^2} - \frac{\ell(\ell+1)}{\rho^2} \right\} u_{j,\ell}(\rho) = 0 \quad (4.11)$$

Solutions for $u_{j,\ell}(\rho)$ involve Gegenbauer polynomials $C_n^{(b)}$, where n is a nonnegative integer and $b > \frac{1}{2}$. They are

$$u_{j,\ell}(\rho) = \mathcal{N}_{j,\ell} \sin^{\ell+1} \chi \cos^{\alpha-1} \chi C_{2j-\ell}^{(\ell+1)}(\cos \chi), \quad (4.12)$$

where $\mathcal{N}_{j,\ell}$ are normalization constants. The angle χ is defined by

$$\tan \chi = \rho, \quad (4.13)$$

and runs from 0 to $\pi/2$. Then the eigenfunctions of $\phi_{j,\ell,m}(\vec{p})$ are

$$\phi_{j,\ell,m}(\vec{p}) = \mathcal{N}_{j,\ell} \sin^\ell \chi \cos^\alpha \chi C_{2j-\ell}^{(\ell+1)}(\cos \chi) Y_m^\ell(\theta, \phi) \quad (4.14)$$

Using the measure (3.7), the norm of $\phi_{j,\ell,m}(\vec{p})$ is finite due to the ultraviolet scale Λ . Furthermore, it is independent of the parameter α , as the α dependence in $\phi_{j,\ell,m}(\vec{p})$ is canceled out by the α dependence in the measure.

The eigenfunctions $\phi_{j,\ell,m}(\vec{p})$ are related to spherical harmonics on S^3 . If we set $\alpha = 0$, they are in fact identical to the spherical harmonics, up to a normalization factor, and are obtainable from $O(4)$ representation matrices.[14] However, their domain is not all of S^3 . To see this we can embed the three-sphere in \mathbb{R}^4 , by defining

$$\begin{aligned} P_1 &= \sin \chi \sin \theta \cos \phi \\ P_2 &= \sin \chi \sin \theta \sin \phi \\ P_3 &= \sin \chi \cos \theta \\ P_4 &= \cos \chi \end{aligned} \quad (4.15)$$

It follows that $P_1^2 + P_2^2 + P_3^2 + P_4^2 = 1$, and like the eigenvalues of the operators \hat{P}_A defined in (3.10), P_A span a hemisphere of S^3 . As before, $P_4 \geq 0$, since $0 \leq \chi \leq \frac{\pi}{2}$. The restriction of the domain of the spherical harmonics to the half-sphere S_+^3 affects their normalization. More significantly, it affects their completeness relations, as we discuss below.

Demanding that $\phi_{j,\ell,m}(\vec{p})$ are orthonormal,

$$\langle \phi_{j,\ell,m} | \phi_{j',\ell',m'} \rangle = \int d\mu(\vec{p}) \phi_{j,\ell,m}(\vec{p})^* \phi_{j',\ell',m'}(\vec{p}) = \delta_{j,j'} \delta_{\ell,\ell'} \delta_{m,m'}, \quad (4.16)$$

leads to the following conditions on the Gegenbauer polynomials

$$\Lambda^3 \mathcal{N}_{j,\ell}^* \mathcal{N}_{j',\ell'} \int_0^{\frac{\pi}{2}} d\chi \sin^{2\ell+2} \chi C_{2j-\ell}^{(\ell+1)}(\cos \chi) C_{2j'-\ell'}^{(\ell'+1)}(\cos \chi) = \delta_{j,j'}, \quad (4.17)$$

using the measure (3.7). Gegenbauer polynomials $\{C_n^{(\ell+1)}(\xi)\}$ are standardly normalized over the domain $-1 \leq \xi \leq 1$, or equivalently $0 \leq \chi \leq \pi$, rather than $0 \leq \chi \leq \pi/2$. The standard normalization condition is

$$\int_0^\pi d\chi \sin^{2\ell+2} \chi C_n^{(\ell+1)}(\cos \chi) C_{n'}^{(\ell+1)}(\cos \chi) = \frac{\pi (n+2\ell+1)!}{2^{2\ell+1} n! (n+\ell+1)(\ell!)^2} \delta_{n,n'} \quad (4.18)$$

In order to relate this to (4.17), we can use the property

$$C_n^{(\ell+1)}(\xi) = (-1)^n C_n^{(\ell+1)}(-\xi) \quad (4.19)$$

Then $\{C_n^{(\ell+1)}(\xi), n \text{ even}\}$ and $\{C_n^{(\ell+1)}(\xi), n \text{ odd}\}$ form two sets of orthogonal polynomials over the half-domain, $0 \leq \xi \leq 1$, or equivalently $0 \leq \chi \leq \pi/2$. From (4.17) and (4.18), the normalization constants are given by

$$|\mathcal{N}_{j,\ell}|^2 = \frac{2^{2\ell+2} (2j+1)(2j-\ell)!(\ell!)^2}{\Lambda^3 \pi (2j+\ell+1)!} \quad (4.20)$$

We note that the set $\{C_n^{(\ell+1)}(\xi), n \text{ even}\}$ is *not* orthogonal to $\{C_n^{(\ell+1)}(\xi), n \text{ odd}\}$ over the half-domain, $0 \leq \xi \leq 1$. Thus there are two distinct sets of orthonormal polynomials. Using $n = 2j - \ell$, they correspond to either j equal to an integer or j equal to a half-integer, and for any given value of ℓ . There are then two distinct sets of orthonormal eigenfunctions $\phi_{j,\ell,m}(\vec{p})$, and two distinct sets of spherical harmonics on S_+^3 . As in the previous subsection, we find that there are two representations of the Snyder algebra, \mathcal{H}_B and \mathcal{H}_F , the former associated with all integer values of j and the latter associated with all half-integer values of j .

Unlike the derivation in subsection 4.1, here we did not a priori make any assumptions about the topology of momentum space, such as (3.19), which identifies opposite points at infinity. The two bases of eigenfunctions which result here are distinguished by their asymptotic properties. Restricting to $\alpha = 0$, we get

$$\phi_{j,\ell,m}(\vec{p}) \sim C_{2j-\ell}^{(\ell+1)}(0) Y_m^\ell(\theta, \phi) \quad \text{as } |\vec{p}| \rightarrow \infty \quad (4.21)$$

From (4.19), $C_{2j-\ell}^{(\ell+1)}(0)$ vanishes when $2j - \ell$ equals an odd integer. Using this and the well known property

$$Y_m^\ell(\pi - \theta, \pi + \phi) = (-1)^\ell Y_m^\ell(\theta, \phi), \quad (4.22)$$

it follows that the asymptotic expression (4.21) has even parity when j is an integer and odd parity when j is a half-integer. Trivial examples of this are the zero angular momentum, parity even, eigenfunctions

$$\phi_{j,0,0}(\vec{p}) = \frac{\sin(1+2j)\chi}{\Lambda^{3/2} \pi \sin \chi}, \quad (4.23)$$

again assuming $\alpha = 0$, which vanish as $|\vec{p}| \rightarrow \infty$ when j is half-integer. In general, wavefunctions Ψ_B in \mathcal{H}_B satisfy

$$\Psi_B(\vec{p}) = \Psi_B(-\vec{p}), \quad \text{as } |\vec{p}| \rightarrow \infty, \quad (4.24)$$

while Ψ_F in \mathcal{H}_F satisfy

$$\Psi_F(\vec{p}) = -\Psi_F(-\vec{p}) , \quad \text{as} \quad |\vec{p}| \rightarrow \infty \quad (4.25)$$

The eigenfunctions (4.14) provide a transform from momentum space to discrete position space, which in this basis is composed of concentric spheres of radii equal to $\sqrt{4j(j+1) - \ell(\ell+1)}/\Lambda$. If $\Psi(\vec{p})$ denotes a wavefunction in the former space and $\Psi_{j,\ell,m}^\circ$ is the corresponding wavefunction on the discrete space, then the transform and its inverse are given by

$$\Psi_{j,\ell,m}^\circ = \int d\mu(\vec{p}) \phi_{j,\ell,m}(\vec{p})^* \Psi(\vec{p}) \quad (4.26)$$

$$\begin{aligned} \Psi(\vec{p}) &= \sum_{j=0,1,2,\dots} \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} \phi_{j,\ell,m}(\vec{p}) \Psi_{j,\ell,m}^\circ \\ &\quad \text{or} \\ &j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \end{aligned} \quad (4.27)$$

The sum over integer j is for Hilbert space \mathcal{H}_B and half-integer j is for \mathcal{H}_F . In Appendix B, we give an example of such a transform, which can be performed exactly.

Finally, in addition to (4.12), there are another set of solutions for $u_{j,\ell}(\rho)$ in (4.11). They can be expressed in terms of hypergeometric functions ${}_2F_1$ according to

$$\frac{\rho^{-\ell}}{(1+\rho^2)^{j+\frac{\alpha}{2}}} {}_2F_1 \left(-j - \frac{\ell}{2} - \frac{1}{2}, -j - \frac{\ell}{2}; \frac{1}{2} - \ell; -\rho^2 \right) \quad (4.28)$$

They lead to a complementary set of eigenfunctions $\{\phi'_{j,\ell,m}(\vec{p})\}$ of $\hat{x}_i \hat{x}_i$, $\hat{L}_i \hat{L}_i$, and \hat{L}_3 , which unlike (4.12), are singular at the origin in momentum space. The singularity is integrable only for the case of zero angular momentum, where the eigenfunction is given by

$$\phi'_{j,0,0}(\vec{p}) = \frac{\cos(1+2j)\chi}{\Lambda^{3/2}\pi \sin \chi} , \quad (4.29)$$

and we again take $\alpha = 0$. The solutions (4.23) and (4.29) comprise the spherically symmetric waves for the system.

4.3 Eigenfunctions of $\hat{A}_i \hat{A}_i$, \hat{x}_3 and \hat{L}_3

In the previous section we found eigenfunctions of the radial coordinate operator. Here we examine eigenfunctions of \hat{x}_3 . More precisely, we study the momentum-dependent basis functions associated with the eigenvectors $\{|j, m_A, m_B\rangle\}$ of Sec 2.2. We denote these basis functions by $\eta_{j,m_A,m_B}(\vec{p}) = \langle \vec{p} | j, m_A, m_B \rangle$. They are obtained from $\phi_{j,\ell,m}(\vec{p})$ in (4.14) by a change of basis

$$\phi_{j,\ell,m}(\vec{p}) = \sum_{m_A=-j}^j \langle j, j; m_A, m - m_A | \ell, m \rangle \eta_{j,m_A,m-m_A}(\vec{p}) , \quad (4.30)$$

where $\langle j, j; m_A, m_B | \ell, m \rangle$ are Clebsch-Gordan coefficients. Expressions for $\eta_{j, m_A, m_B}(\vec{p})$ can be given in terms of the cylindrical variables R_p, ϕ_p, p_3 , [where $p_1 = R_p \cos \phi_p$, $p_2 = R_p \sin \phi_p$ and $0 \leq \phi_p < 2\pi$]. The eigenfunctions have the general form

$$\eta_{j, m_A, m_B}(\vec{p}) = f_{j, m_A, m_B}(R_p, p_3) e^{i(m_A + m_B)\phi_p}, \quad (4.31)$$

where $f_{j, m_A, m_B}(R_p, p_3)$ are eigenfunctions of $\hat{A}_i \hat{A}_i$ and \hat{x}_3 . We can obtain the precise form of $f_{j, m_A, m_B}(R_p, p_3)$ for the special case of zero angular momentum in the third direction. Then, $\eta_{j, m_A, -m_A}(\vec{p}) = f_{j, m_A, -m_A}(R_p, p_3)$, corresponding to plane waves in the third direction. From the eigenvalue equation

$$\hat{x}_3 \eta_{j, m_A, -m_A}(\vec{p}) = 2m_A \eta_{j, m_A, -m_A}(\vec{p}), \quad (4.32)$$

we get

$$\eta_{j, m_A, -m_A}(\vec{p}) = e^{-2i m_A \tan^{-1}(\frac{p_3}{\Lambda})} \mathcal{F}_{j, m_A}\left(\frac{\vec{p}^2 + \Lambda^2}{p_3^2 + \Lambda^2}\right), \quad (4.33)$$

where we used the differential representation (3.6) with $\alpha = 0$. The functions $\mathcal{F}_{j, m_A}(\zeta)$ are determined from the remaining eigenvalue equation

$$\hat{A}_i \hat{A}_i \eta_{j, m_A, -m_A}(\vec{p}) = j(j+1) \eta_{j, m_A, -m_A}(\vec{p}), \quad (4.34)$$

which leads to

$$\zeta^2 \frac{d}{d\zeta} \left((\zeta - 1) \frac{d}{d\zeta} \right) \mathcal{F}_{j, m_A}(\zeta) + (j^2 + j - m_A^2 \zeta) \mathcal{F}_{j, m_A}(\zeta) = 0 \quad (4.35)$$

Solutions for $\mathcal{F}_{j, m_A}(\zeta)$ can be expressed in terms of hypergeometric functions. Up to a normalization, $\mathcal{F}_{j, m_A}(\zeta)$ is

$$\zeta^{-j} {}_2F_1(-j - m_A, m_A - j; -2j; \zeta) \quad (4.36)$$

Additional solutions are

$$\zeta^{j+1} {}_2F_1(j - m_A + 1, j + m_A + 1; 2j + 2; \zeta), \quad (4.37)$$

which are singular at $\zeta = 1$. Using $\zeta = (\vec{p}^2 + \Lambda^2)/(p_3^2 + \Lambda^2)$, the latter are divergent along p_3 -axis.

The eigenfunctions $\eta_{j, m_A, m_B}(\vec{p})$ provide a transform from momentum space to the spatial lattice (with only x_3 determined). If $\Psi(\vec{p})$ is a wavefunction in the former space and $\Psi_{j, m_A, m_B}^\#$ is a wavefunction on the lattice, then they are related by

$$\Psi_{j, m_A, m_B}^\# = \int d\mu(\vec{p}) \eta_{j, m_A, m_B}(\vec{p})^* \Psi(\vec{p}) \quad (4.38)$$

$$\begin{aligned} \Psi(\vec{p}) &= \sum_{j=0,1,2,\dots} \sum_{m_A, m_B=-j}^j \eta_{j, m_A, m_B}(\vec{p}) \Psi_{j, m_A, m_B}^\# \\ &\text{or} \\ j &= \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \end{aligned} \quad (4.39)$$

Once again, the sum over integer j is for Hilbert space \mathcal{H}_B and half-integer j is for \mathcal{H}_F .

5 Rotations and Translations

It is now straightforward to write down the unitary action of the rotation and translation group on Snyder space. They are generated by \hat{L}_i and \hat{p}_i , respectively. The latter is not responsible for discrete translations from one point on the spatial lattice to a neighboring point. Rather, \hat{p}_i generate continuous translations in the basis $\{|\vec{q}\rangle\}$ which diagonalizes the operators \hat{q}_i conjugate to \hat{p}_i . [Cf. eq. (3.2).] Differential representations associated with this basis were given in eq. (3.9).

In the basis $\{|j, \ell, m\rangle\}$, Snyder space corresponds to the set of concentric spheres with radii equal to $\sqrt{4j(j+1) - \ell(\ell+1)}/\Lambda$. $m = -\ell, \dots, \ell$ is a degeneracy index. A rotation by $\vec{\theta}$ is given by

$$e^{i\theta_i \hat{L}_i} |j, \ell, m\rangle = \sum_{m'=-\ell}^{\ell} D_{m',m}^{\ell}(\vec{\theta}) |j, \ell, m'\rangle, \quad (5.1)$$

where D^{ℓ} are $SO(3)$ matrix representations, $D_{m',m}^{\ell}(\vec{\theta}) = \langle \ell, m' | e^{i\theta_i \hat{L}_i} | \ell, m \rangle$ and $| \ell, m \rangle$ are the usual angular momentum eigenstates. A translation by \vec{a} is given by

$$\begin{aligned} e^{ia_i \hat{p}_i} |j, \ell, m\rangle &= \\ & \sum_{j'=0,1,2,\dots} \sum_{\ell'=0}^{2j'} \sum_{m'=-\ell'}^{\ell'} \int d\mu(\vec{p}) \quad |j', \ell', m'\rangle e^{ia_i p_i} \phi_{j', \ell', m'}(\vec{p})^* \phi_{j, \ell, m}(\vec{p}), \\ & \text{or} \\ & j' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \end{aligned} \quad (5.2)$$

and one sums over integer (half-integer) values of j' when j is integer (half-integer), because the translation operator does not take states out of the Hilbert space.

In the basis $\{|j, m_A, m_B\rangle\}$, Snyder space corresponds to a lattice with the eigenvalues of the third coordinate given by $(m_A - m_B)/\Lambda$. Using the Clebsch-Gordan coefficients, rotations act according to

$$\begin{aligned} e^{i\theta_i \hat{L}_i} |j, m_A, m_B\rangle &= \\ & \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} \sum_{m'_A, m'_B=-j}^j |j, m'_A, m'_B\rangle \langle j, j; m'_A, m'_B | \ell, m \rangle \times \\ & \langle j, j; m_A, m_B | \ell, m_A + m_B \rangle^* D_{m, m_A + m_B}^{\ell}(\vec{\theta}), \end{aligned} \quad (5.3)$$

while translations are given by

$$e^{ia_i \hat{p}_i} |j, m_A, m_B\rangle =$$

$$\begin{aligned}
& \sum_{j'=0,1,2,\dots} \sum_{m'_A, m'_B=-j'}^{j'} \int d\mu(\vec{p}) \quad |j', m'_A, m'_B\rangle e^{ia_i p_i} \eta_{j', m'_A, m'_B}(\vec{p})^* \eta_{j, m_A, m_B}(\vec{p}), \\
& \text{or} \\
& j' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots
\end{aligned} \tag{5.4}$$

Again, one sums over integer (half-integer) values of j' when j is integer (half-integer). As stated above, $e^{ia_i \hat{p}_i}$ does not, in general, map one point on the spatial lattice to another point. To hop from one eigenvalue of \hat{x}_3 to another, one needs an operator that maps (m_A, m_B) to (m'_A, m'_B) , with $m_A - m_B \neq m'_A - m'_B$. Such an operator need not be unitary; Examples are the raising and lowering operators $\hat{A}_1 \pm i\hat{A}_2$ and $\hat{B}_1 \pm i\hat{B}_2$.

6 Discussion

We have obtained basis functions for the Snyder algebra and have argued that they span two distinct Hilbert spaces, \mathcal{H}_B , and \mathcal{H}_F . The Hilbert spaces are associated with different lattice structures. Two different lattice structures were also noted in the case of particles in $2+1$ gravity and were referred to as bosonic and fermionic.[2] In our case, the lattices are distinguished by an $SU(2)$ quantum number, which takes integer values for \mathcal{H}_B and half-integer values for \mathcal{H}_F . In analogy with spin $SU(2)$, there may exist a lattice-statistics connection, whereby bosons are associated with \mathcal{H}_B and fermions are associated with \mathcal{H}_F . Also, a supersymmetric generalization of this system may lead to \mathcal{H}_B and \mathcal{H}_F coexisting in a graded space.

We showed that rotations and translations can be unitarily implemented on the lattice structures. In generalizing to the Poincaré group for relativistic systems (or Galilei group for nonrelativistic systems), we need to re-introduce the notions of time and energy, which were associated with independent generators in Snyder's algebra.[1] Snyder's time operator has a continuous spectrum, and its commutation relations were postulated in order to maintain Lorentz invariance. Time operators can be a source of additional conceptual problems in quantum mechanics. Alternatively, one can introduce the notion of time through particle dynamics. Previous articles have addressed the issue of particle dynamics in Snyder space-time,[17], [5] a number of which have focused on the problem of deriving the Snyder algebra starting from an action principle.[18],[19],[20],[21],[11] In a forthcoming work,[4] we shall examine the motion of a relativistic particle on the three-dimensional Euclidean Snyder space discussed here. We shall show that, 'time' can be either identified with a real number λ which parametrizes the particle evolution, or it can be some function of λ and the fundamental operators, \hat{x}_i and \hat{p}_i . In the latter, time is an λ -dependent operator whose commutation relations can be derived from (2.1). The resulting algebra is not Lorentz covariant. Nevertheless, the time operator can be used to construct Lorentz boost operators, from which we can then recover the Lorentz

algebra. Upon including the free particle Hamiltonian operator, one recovers the Poincaré algebra, and so the full Poincaré group can be unitarily implemented on the lattice.

Finally, there have been various attempts to write down field theory on Snyder space with the goal of having a consistent Lorentz invariant noncommutative quantum field theory[15],[16]. The approach considered so far is to try to write down star product representations of the Snyder algebra on a commutative space-time manifold. The proposed star products have introduced some confusion, in that they are either nonassociative or lead to a deformation of the Poincaré symmetries.[15],[16] Since Snyder space is, in fact, a lattice, a more appropriate way to proceed with the second quantization of the theory may be to consider lattice field theory. Unlike usual lattice field theory, here fields can be defined in only one direction of the spatial lattice; this would correspond, for example, to the radial direction if we use the basis $\{|j, \ell, m \rangle_\circ\}$, or the third Cartesian direction if we use the basis $\{|j, m_A, m_B \rangle\}$. Assuming the latter, fields are a function of quantum numbers j, m_A and m_B , and some continuous evolution parameter, say λ . It can also depend on spin quantum numbers. (As stated above, the system, here expressed in terms of fields, should carry representations of the Poincaré group.) In the simplest case of a real scalar field $\Phi(\lambda)_{j, m_A, m_B}$, we can write

$$\Phi(\lambda)_{j, m_A, m_B} = \int d\mu(\vec{p}) \left(\mathbf{a}(\vec{p}) \eta_{j, m_A, m_B}(\vec{p}) e^{-i\lambda H} + \mathbf{a}(\vec{p})^\dagger \eta_{j, m_A, m_B}(\vec{p})^* e^{i\lambda H} \right), \quad (6.1)$$

where H generates evolution in λ , and we introduced particle creation (annihilation) operators $\mathbf{a}(\vec{p})^\dagger$ ($\mathbf{a}(\vec{p})$). The field action $\mathcal{S}[\Phi]$ involves an integral over λ , as well as a sum over the lattice,

$$\mathcal{S}[\Phi] = \int d\lambda \sum_{\mathcal{H}_B \text{ or } \mathcal{H}_F} \mathcal{L}[\Phi(\lambda)], \quad (6.2)$$

where $\mathcal{L}[\Phi(\lambda)]$ is the field Lagrangian. If the lattice-statistics connection mentioned above applies, then the sum is over the Hilbert space \mathcal{H}_B for bosonic fields and \mathcal{H}_F for fermionic fields. A nontrivial problem is then to find a Lagrangian on the lattice which gives relativistic invariant dynamics and reduces to a familiar model in the limit of zero lattice spacing $\Lambda \rightarrow \infty$.

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Appendix A Sample wavepackets

Here we construct normalizable wavepackets in momentum space which can be explicitly transformed to the discrete position space. The transformed wavepackets are functions on the set of concentric spheres with radii $\sqrt{4j(j+1) - \ell(\ell+1)}/\Lambda$. For the examples which follow, we can carry out the integral in (4.26) analytically. The transformation can be performed

explicitly thanks to the following well known identity for Gegenbauer polynomials:

$$\frac{1}{(1 - 2\kappa \cos \chi + \kappa^2)^{\ell+1}} = \sum_{n=0}^{\infty} C_n^{(\ell+1)}(\cos \chi) \kappa^n, \quad (\text{A.1})$$

where κ is in general a complex parameter with $|\kappa| < 1$. From subsection 4.2, the Gegenbauer polynomials which appear in the eigenfunctions $\phi_{j,\ell,m}(\vec{p})$ are restricted to the half-interval, $0 \leq \chi \leq \pi/2$, and they form two orthogonal sets, those with n even or n odd. From (4.14), n should be identified with $2j - \ell$. Using the property (4.19), one can obtain two additional identities from (A.1) which involve sums either over all even n or all odd n :

$$\begin{aligned} f_{\pm}^{(\ell,\kappa)}(\chi) &= \frac{1}{(1 + 2\kappa \cos \chi + \kappa^2)^{\ell+1}} \pm \frac{1}{(1 - 2\kappa \cos \chi + \kappa^2)^{\ell+1}} \\ &= \sum_{n=\begin{cases} 0,2,4,\dots \\ 1,3,5,\dots \end{cases}} C_n^{(\ell+1)}(\cos \chi) \kappa^n \end{aligned} \quad (\text{A.2})$$

The function $f_+^{(\ell,\kappa)}$ is not normalizable, i.e., $\int_0^{\pi/2} d\chi \sin^{2\ell+2} \chi |f_+^{(\ell,\kappa)}(\chi)|^2$ is not finite, and so we restrict to $f_-^{(\ell,\kappa)}$ and sum over odd n . For any given ℓ, m and κ , we can then construct the normalizable momentum-dependent wavepacket

$$\Psi^{(\ell,m,\kappa)}(\vec{p}) = \mathcal{A}(\kappa)_{\ell,m} f_-^{(\ell,\kappa)}(\chi) \sin^{\ell} \chi Y_m^{\ell}(\theta, \phi), \quad (\text{A.3})$$

where the momentum is defined in terms of angles χ, θ and ϕ according to (4.15) and $\mathcal{A}(\kappa)_{\ell,m}$ is a normalization factor. It can be expanded in terms of basis functions $\phi_{j,\ell,m}$ defined in (4.14),

$$\Psi^{(\ell,m,\kappa)}(\vec{p}) = \mathcal{A}(\kappa)_{\ell,m} \sum_{2j-\ell=1,3,5,\dots} \frac{\kappa^{2j-\ell}}{\mathcal{N}_{j,\ell}} \phi_{j,\ell,m}(\vec{p}), \quad (\text{A.4})$$

and we have chosen $\alpha = 0$ for convenience. In comparing with (4.27), the corresponding wavefunction $\Psi^{\circ(\ell,m,\kappa)}$ in position space is given by

$$\Psi_{j,\ell',m'}^{\circ(\ell,m,\kappa)} = \frac{\mathcal{A}(\kappa)_{\ell,m}}{\mathcal{N}_{j,\ell}} \kappa^{2j-\ell} \delta_{\ell,\ell'} \delta_{m,m'} \quad (\text{A.5})$$

As an example, we can consider spherically symmetric wavepacket, $\ell = m = 0$,

$$\Psi^{(0,0,\kappa)}(\vec{p}) = \frac{\sqrt{1-\kappa^4}}{(\Lambda\pi)^{3/4}} \frac{2 \cos \chi}{1 - 2\kappa^2 \cos 2\chi + \kappa^4} \quad (\text{A.6})$$

Its transform to position space is

$$\Psi_{j,\ell',m'}^{\circ(0,0,\kappa)} = \frac{\sqrt{1-\kappa^4}}{2} \kappa^{2j-1} \delta_{\ell',0} \delta_{m',0} \quad (\text{A.7})$$

From (A.6) and the differential representation (3.6), one can compute the mean values of \vec{x}^2 and \vec{p}^2 for these wavepackets. The result is

$$\langle \vec{x}^2 \rangle_{(0,0,\kappa)} = \frac{3 + 6\kappa^4 - \kappa^8}{\Lambda^2(1 - \kappa^4)^2}$$

$$\langle \bar{p}^2 \rangle_{(0,0,\kappa)} = \Lambda^2 \frac{(1 - \kappa^2)(3 + \kappa^2)}{(1 + \kappa^2)^2} \quad (\text{A.8})$$

Appendix B Comparison with the Hydrogen atom

There are several similarities, along with some interesting differences, between the three-dimensional (Euclidean) Snyder algebra (2.1) and the algebra of the hydrogen atom, which we point out in this appendix. Both systems contain an $so(4)$ subalgebra and the representations which appear in the quantum theory are infinite-dimensional and reducible. However, their action on momentum space is different for the two systems.

We recall some features of the $so(4)$ algebra for the hydrogen atom, first in the classical theory. The group is generated by the angular momentum L_i^{H} and

$$x_i^{\text{H}} = \frac{1}{\sqrt{-H}} K_i^{\text{H}}, \quad (\text{B.1})$$

where H is the Hamiltonian

$$H = \frac{p_i^{\text{H}} p_i^{\text{H}}}{2\mu} - \frac{e^2}{|q^{\text{H}}|}, \quad (\text{B.2})$$

and K_i^{H} is the Runge-Lenz vector

$$K_i^{\text{H}} = \frac{1}{\mu e^2} \epsilon_{ijk} L_j^{\text{H}} p_k^{\text{H}} + \frac{q_i^{\text{H}}}{|q^{\text{H}}|} \quad (\text{B.3})$$

q_i^{H} and p_i^{H} are canonically conjugate phase space variables $\{q_i^{\text{H}}, p_j^{\text{H}}\} = \delta_{ij}$, the angular momentum is as usual given by $L_i^{\text{H}} = \epsilon_{ijk} q_j^{\text{H}} p_k^{\text{H}}$ and we restrict to bound states $H < 0$. μ and e are, respectively, the reduced mass and charge. The Poisson brackets for L_i^{H} and x_i^{H} are given by

$$\begin{aligned} \{x_i^{\text{H}}, x_j^{\text{H}}\} &= \frac{2}{\mu e^4} \epsilon_{ijk} L_k^{\text{H}} \\ \{x_i^{\text{H}}, L_j^{\text{H}}\} &= \epsilon_{ijk} x_k^{\text{H}} \\ \{L_i^{\text{H}}, L_j^{\text{H}}\} &= \epsilon_{ijk} L_k^{\text{H}} \end{aligned} \quad (\text{B.4})$$

In comparing with (2.1) and (2.2), we see that here $\sqrt{\frac{\mu}{2}} e^2$ plays the role of the deformation parameter Λ in the Snyder algebra. Also, like in (2.7), the scalar product of L_i^{H} and x_i^{H} vanishes, and so just as with the Snyder algebra, there is only one independent quadratic invariant of $SO(4)$.[¶] The quadratic Casimir is $\left(L_i^{\text{H}} \pm \sqrt{\frac{\mu}{2}} e^2 x_i^{\text{H}}\right)^2 = L_i^{\text{H}} L_i^{\text{H}} + \frac{\mu e^4}{2} x_i^{\text{H}} x_i^{\text{H}}$, and its eigenvalues $\hbar^2 j(j+1)$ label the irreducible representations of the $SO(4)$ algebra occurring in the quantum theory. The index j is related to the principal quantum number n_p according to $n_p = 2j + 1$. [6]

[¶]We thank Tom Kephart for pointing this out.

In the Snyder algebra, the action of the $SO(4)$ generators on momentum is given by the commutation relations (2.1) and (2.2). One gets a more complicated action of $SO(4)$ on momentum in the hydrogen atom system. The Poisson brackets of the generators with the momenta lead to a mixing with the position variables $q_i^{\mathbb{H}}$,

$$\begin{aligned}\{x_i^{\mathbb{H}}, p_j^{\mathbb{H}}\} &= \frac{1}{\sqrt{-H}} \left(\frac{p_i^{\mathbb{H}} p_j^{\mathbb{H}} - \delta_{ij} |p^{\mathbb{H}}|^2}{\mu e^2} - \frac{q_i^{\mathbb{H}} q_j^{\mathbb{H}} - \delta_{ij} |q^{\mathbb{H}}|^2}{|q^{\mathbb{H}}|^3} + \frac{e^2 x_i^{\mathbb{H}} q_j^{\mathbb{H}}}{2\sqrt{-H} |q^{\mathbb{H}}|^3} \right) \\ \{L_i^{\mathbb{H}}, p_j^{\mathbb{H}}\} &= \epsilon_{ijk} p_k^{\mathbb{H}}\end{aligned}\tag{B.5}$$

The energy eigenfunctions for the hydrogen atom in momentum space were found long ago by Podolsky and Pauling[7] and by Fock[8]. They are spherical harmonics on S^3 , and so are given by the expressions (4.14) [with $\alpha = 0$]. The three-sphere is not obtained in this case using the map (3.10). Rather, it is obtained by an inverse stereographic projection of momentum space. Here the four-dimensional embedding coordinates P_A , $A = 1, 2, 3, 4$, $P_A P_A = 1$, associated with any given bound state energy E_n are given by[6]

$$P_i = \frac{2p_i^{\mathbb{H}} \sqrt{-2mE_n}}{|p^{\mathbb{H}}|^2 - 2mE_n} \quad P_4 = \frac{|p^{\mathbb{H}}|^2 + 2mE_n}{|p^{\mathbb{H}}|^2 - 2mE_n}\tag{B.6}$$

Unlike (3.10), this map takes \mathbb{R}^3 to all of S^3 . Setting $P_4 = \cos \chi$ as before, χ now ranges from 0 to π , rather than 0 to $\pi/2$, as with the eigenfunctions (4.14) for the Snyder algebra. We argued in section 4 that there were two distinct sets of orthonormal eigenfunctions for the Snyder algebra, and that they correspond to j either all integers or all half-integers. Here, however, since the domain is $0 \leq \chi \leq \pi$ for the hydrogen atom system, the Gegenbauer polynomials $\{C_n^{(\ell+1)}(\cos \chi), n = 0, 1, 2, \dots\}$ forms one complete orthonormal set, and using $n = 2j - \ell$, the spherical harmonics with both integer and half-integer values for j span the Hilbert space. Unlike in the subsection 4.1, the basis functions now consist of all irreducible matrix representations of $SU(2)$, $\{D^j(g), j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$, which allows the principal quantum number $n_p = 2j + 1$ to be any positive integer.

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